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The Navier–Stokes Limit of the Stationary Boltzmann Equation for Hard Potentials

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In this paper we extend recent results on the hydrodynamic Navier–Stokes limit of the stationary Boltzmann equation for the flow of a gas of hard spheres in a channel in the presence of an external force to the case of a hard intermolecular potential with Grad angular cutoff. We prove the convergence of the solution, for small Knudsen numbers, to the Maxwellian with parameters solving the corresponding Navier–Stokes equation. In the present case we only get polynomial decay of the solution for large velocities, instead of the exponential decay which holds for hard spheres.

KEY WORDS: Hydrodynamic limit; stationary Navier–Stokes equations; kinetic theory.

1. INTRODUCTION

The macroscopic stationary behavior of a Boltzmann gas has been studied recently[1,2] under special symmetry conditions, corresponding to a one-dimensional flow between infinite parallel plates at fixed temperatures maintained by a constant external force parallel to the plates. As a result, it was proved that the solution of the stationary Boltzmann equation converges, as the mean free path goes to zero, to a local Maxwellian with parameters satisfying the stationary Navier–Stokes equations.

We refer to refs. 1 and 2 for motivations and general framework and recall that the results obtained there were restricted to the case of hard spheres. But the transport coefficients in several situations are not correctly approximated by the ones corresponding to hard spheres. Then it is natural

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to try to extend the results of refs. 1 and 2 to more realistic cross sections. To remove the restriction to hard spheres, however, we need (at least for nonvanishing external forces) nonobvious modifications of the method presented in refs. 1 and 2. The reason for this is the following. The natural space to deal with this problem is a space of functions decaying exponentially for large velocities. In this framework, when there is an external force, some divergences arise due to the differentiation of the distribution function with respect to the velocity. The quadratic form associated with the linearized Boltzmann operator in the case of hard spheres is strong enough to dominate such diverging terms. In the case of hard intermolecular potentials the quadratic form is smaller than for hard spheres and is no longer sufficient for our purposes. On the other hand, if we consider polynomial decay for high velocities, the terms produced by the velocity derivatives are uniformly bounded.

In this paper we take advantage of this remark to deal with hard potentials. Namely, we give up the exponential decay of the solution, switching to an algebraic framework. The general structure of the method is similar to the one used in refs. 1 and 2. We look for a solution in the form of a truncated bulk + boundary layer expansion with a remainder. The expansion of the solution is the same as in ref. 1, while modifications are necessary in the estimate of the remainder. This estimate is based on a decomposition into a low-velocity part and a high-velocity part. Of course the first one is not seriously affected by the choice of the decay (exponential or algebraic). The most serious changes are for the high-velocity part. In fact, to deal with it we need to estimate the collision operator in L_2 and L_∞ norms with polynomial weights. L_∞-estimates of this kind were previously used in ref. 3 and can be adapted to the present setup. L_2-estimates are obtained in this paper using similar ideas. They depend on the degree of the polynomial weight, in the sense that the bounds contain a *good* part and a *bad* part. This last part is multiplied by a small factor when the polynomial has sufficiently high degree. Using such estimates, we can show that, if we consider sufficiently fast polynomial decay, we can bound the high-velocity part of the remainder in a suitable norm. In this way we obtain a solution of the stationary Boltzmann equation converging to the Maxwellian with parameters satisfying the stationary Navier–Stokes equations for the flow between parallel plates at fixed temperatures, subject to an external force parallel to the plates.

The paper is organized as follows: in Section 2 we briefly state the problem and the result. In Section 3 we sketch the method of solution, pointing out the differences from ref. 1. Finally, in Section 4 we give the estimate for the high-velocity part. We will use several notations introduced in ref. 1.

2. MAIN RESULT

We consider the stationary flow of a Boltzmann gas in a channel perpendicular to the y axis, of size $2\varepsilon^{-1}$ in microscopic units, with infinite parallel walls at given temperatures $T_+ \geqslant T_-$, subject to a constant force parallel to the walls, say along the x axis, with intensity $\varepsilon^2 F$. This choice of the size of the force is dictated by the aim that there is a stationary limiting solution. The arguments to justify it are given in ref. 1 and we do not repeat them. Assuming the space dependence of the distribution function f only on the variable y, we obtain the following boundary value problem corresponding to this flow, after rescaling the space variable:

$$\begin{aligned}
&v_y \frac{\partial f}{\partial y} + \varepsilon F \frac{\partial f}{\partial v_x} = \frac{1}{\varepsilon} Q(f, f), \qquad (y, v) \in (-1, 1) \times \mathbb{R}^3 \\
&f(-1, v) = \alpha_- \bar{M}_-(v) \qquad \text{for} \quad v_y > 0 \\
&f(1, v) = \alpha_+ \bar{M}_+(v) \qquad \text{for} \quad v_y < 0 \\
&\int_{-1}^{1} dy \int_{\mathbb{R}^3} dv\, f(y, v) = m
\end{aligned} \tag{2.1}$$

The boundary conditions correspond to reflecting particles hitting the boundaries randomly with Maxwellian distributions

$$\bar{M}_\pm(v) = \frac{1}{2\pi T_\pm^2} e^{-[v - U_\pm]^2/2T_\pm} \tag{2.2}$$

normalized so that

$$\int_{v_y \lessgtr 0} |v_y|\, \bar{M}_\pm(v)\, dv = 1$$

The coefficients $\alpha_\pm$ are given by

$$\alpha_\pm = \pm \int_{v_y \gtrless 0} v_y f(\pm 1, v)\, dv \tag{2.3}$$

and are determined so that the net mass current at the walls vanishes. $U_\pm$ are two vectors parallel to the walls, representing the translation velocities of the walls. Finally, $Q(f, g)$ is the Boltzmann collision operator given by

$$\begin{aligned}
Q(f, g)(v) = \tfrac{1}{2} \int_{\mathbb{R}^3} dv_1 \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta\, B(|v - v_1|, \theta) \\
\times [f(v')\, g(v_1') + g(v')\, f(v_1') - f(v)\, g(v_1) - g(v)\, f(v_1)]
\end{aligned} \tag{2.4}$$

with $B(|v-v_1|, \theta)$ the scattering cross section and v, v_1, v', v'_1 the incoming and outgoing velocities in a collision with impact parameter $n \in S^2$ corresponding to the polar and azimuthal angles θ and φ and polar axis parallel to $v-v_1$. We assume *hard* molecular interactions with force depending on the interparticle distance r as r^{-k}, $k \geqslant 5$, with Grad *angular cutoff.*[4] This implies that

$$B(|v-v_1|, \theta) \leqslant |v-v_1|^{\beta} h(\theta) \tag{2.5}$$

with $\beta=(k-5)/(k-1)$ and $h(\theta)$ a bounded continuous function. The limiting case $k=\infty$, $\beta=1$ is the hard-sphere interaction discussed in refs. 1 and 2. In this paper we will deal with finite k's and hence with $0 \leqslant \beta < 1$. Of course the arguments are valid also for $\beta=1$, but in this case they provide weaker results than those already obtained in refs. 1 and 2.

We will use the following norms, for integers $p>0$:

$$|f|_p = \sup_{y \in [-1,1]} \sup_{v \in \mathbb{R}^3} (1+v^2)^{p/2} |f(y, v)| \tag{2.6}$$

Let $q=\max\{|F|, |T_+ - T_-|, |U_+ - U_-|\}$. Our result is summarized as follows.

Theorem 2.1. There are positive ε_0, p_0, and q_0 and a constant c such that for $\varepsilon<\varepsilon_0$, $q<q_0$, and $p>p_0$ there is a solution f^{ε} to the boundary value problem (2.1) such that, denoting by M the Maxwellian,

$$M(y, v) = \frac{\rho(y)}{[2\pi T(y)]^{3/2}} \exp\left[-\frac{[(v_x-u(y))^2 + v_y^2 + (v_z - w(y))^2]}{2T(y)}\right] \tag{2.7}$$

then

$$|f^{\varepsilon} - M|_p \leqslant c\varepsilon \tag{2.8}$$

provided that ρ, u, w, and T are the density, the nonvanishing components of the velocity field, and the temperature solving the Navier–Stokes boundary value problem

$$\frac{d}{dy}(\rho T)=0, \qquad \frac{d}{dy}\left(\eta(T)\frac{du}{dy}\right)+\rho F=0, \qquad \frac{d}{dy}\left(\eta(T)\frac{dw}{dy}\right)=0$$

$$\frac{d}{dy}\left(\kappa(T)\frac{dT}{dy}\right)+\eta(T)\left[\left(\frac{du}{dy}\right)^2+\left(\frac{dw}{dy}\right)^2\right]=0 \tag{2.9}$$

$$u(\pm 1)=U_{\pm,x}, \qquad w(\pm 1)=U_{\pm,z}, \qquad T(\pm 1)=T_{\pm}$$

with $\eta(T)$ and $\kappa(T)$ the viscosity coefficient and the heat conductivity, respectively, corresponding to the collision cross section $B(|v-v_1|, \theta)$. Furthermore, the solution is unique in a suitable class which will be specified later.

Remark. The bound (2.8) in terms of the norm (2.6) implies just polynomial decay of the solution f^ε, a much weaker result than the one obtained for hard spheres in refs. 1 and 2. This is a consequence of the method adopted here and it is difficult to get a better decay with the present approach based on a decomposition into low and high velocities. We note, however, that in the case $F=0$ the method used in refs. 1 and 2 works for hard potentials and exponential decay follows for any $\beta \in [0, 1]$.

3. OUTLINE OF THE PROOF

The proof of Theorem 2.1 is based on a bulk + boundary layer expansion with the remainder. The present setup only affects the remainder part, hence most of this section is taken from ref. 1, and we refer to it for the details. We write f^ε as

$$f^\varepsilon = M + \sum_{n=1}^{6} f_n + \varepsilon^3 f_R \tag{3.1}$$

with M the Maxwellian (2.7) with parameters satisfying (2.9). Moreover, $f_n = B_n + b_n^+ + b_n^-$ with B_n and $b_n^\pm$ the solutions of the equations

$$\begin{aligned} v_y \frac{\partial B_{n-1}}{\partial y} + F \frac{\partial B_{n-2}}{\partial v_x} \\ = \mathscr{L} B_n + \sum_{k, m \geqslant 1, k+m=n} Q(B_k, B_m) \end{aligned} \tag{3.2}$$

$$\begin{aligned} v_y \frac{\partial}{\partial y^\pm} b_n^\pm + F \frac{\partial}{\partial v_x} b_{n-2}^\pm \\ = \mathscr{L}_\pm b_n^\pm + 2Q(\Delta M_\pm, b_{n-1}^\pm) \\ + \sum_{i,j \geqslant 1, i+j=n} [2Q(B_i, b_j^\pm) + Q(b_i^\pm, b_j^\pm) + Q(b_i^\mp, b_j^\pm)] \end{aligned} \tag{3.3}$$

$$f_n(\pm 1, v) = \alpha_n^\pm \bar{M}_\pm(v) + \gamma_{n,\varepsilon}^\pm(v)$$

$$v_y \lessgtr 0, \qquad \alpha_n^\pm = \pm \int_{v_y \gtrless 0} dv\, v_y f_n(\pm 1, v)$$

Here $B_0 = M$, $B_{-1} = 0$, $b_0^{\pm} = b_{-1}^{\pm} = 0$. The $y^{\pm}$ are the space variables scaled as $y^{\pm} = \varepsilon^{-1}(1 \pm y)$, $M_{\pm}$ is the Maxwellian (2.7) evaluated in ± 1, $\gamma_{n,\varepsilon}^{\pm} = b_n^{\mp}(2\varepsilon^{-1}, v)$, and

$$\mathscr{L}f = 2Q(M, f), \qquad \mathscr{L}_{\pm} f = 2Q(M_{\pm}, f) \tag{3.4}$$

The remainder has to solve the boundary value problem

$$v_y \frac{\partial f_R}{\partial y} + \varepsilon F \frac{\partial f_R}{\partial v_x} = \frac{1}{\varepsilon} \mathscr{L} f_R + \mathscr{L}^1 f_R + \varepsilon^2 Q(f_R, f_R) + \varepsilon^3 A$$

$$f_R(\pm 1, v) = \alpha_R^{\pm} \bar{M}_{\pm}(v) - \sum_{n=1}^{6} \varepsilon^{n-3} \gamma_{n,\varepsilon}^{\pm} \qquad \text{for} \quad v_y \lessgtr 0 \tag{3.5}$$

$$\int_{\mathbb{R}^3} dv\, v_y f_R(y, v) = 0, \qquad \int_{-1}^{1} dy \int_{\mathbb{R}^3} dv\, f_R(y, v) = 0$$

with $\mathscr{L}^1 f_R = 2Q(\sum_{n=1}^{6} \varepsilon^{n-1} f_n, f_R)$, $\alpha_R^{\pm}$ to be chosen to satisfy $(3.5)_3$, $\Delta M_{\pm} = \varepsilon^{-1}(M - M_{\pm})$, and

$$A = -\left[v_y \frac{\partial B_6}{\partial y} + \varepsilon F \frac{\partial f_6}{\partial v_x} + F \frac{\partial f_5}{\partial v_x} \right] + [2Q(\Delta M_+, b_6^+) + 2Q(\Delta M_-, b_6^-)]$$
$$+ \sum_{1 \leqslant k,\, m \leqslant 6,\, k+m \geqslant 7} \varepsilon^{k+m-7} Q(f_k, f_m) \tag{3.6}$$

The properties of the f_n are summarized in the following proposition, which is taken from ref. 1.

Proposition 3.1. Let q be a sufficiently small. Then there are unique smooth functions ρ, T, u, and w satisfying (2.9), with derivatives of order k bounded by $C_k q$. Moreover, it is possible to determine uniquely the functions B_n and $b_n^{\pm}$, $n = 1,..., 6$, satisfying (3.2) and (3.3) so that $f_n = B_n + b_n^+ + b_n^-$ satisfies $\int_{-1}^{1} dy \int_{\mathbb{R}^3} dv\, f_n = 0$ and the condition $\int_{\mathbb{R}^3} dv\, v_y f_n = 0$ for any $y \in [-1, 1]$. Furthermore, for any positive integer r there is a constant c such that

$$|M^{-1/2} B_n|_r < cq$$
$$|M_{\pm}^{-1/2} b_n^{\pm}(\varepsilon^{-1}(1 \mp y)) \exp[-\sigma \varepsilon^{-1}(1 \mp y)]|_r < cq \tag{3.7}$$

for some constant $\sigma > 0$. Finally, the function A in (3.6) satisfies

$$\int_{\mathbb{R}^3} dv\, A = 0 \quad \text{for} \quad y \in [-1, 1], \qquad \left| A \exp\left(-\frac{1}{4\vartheta} v^2\right) \right|_r < cq \tag{3.8}$$

for any $\vartheta > \sup_{y \in [-1, 1]} T(y)$.

The solution f_R of (3.5) is sought in the form $f_R = I(R)\,M + R$ with $I(R) = -m^{-1} \int_{-1}^{1} dy \int_{\mathbb{R}^3} dv\, R(y, v)$. After choosing $\alpha_R^- = (T_-/2\pi)^{1/2} \rho_-^1\, I(R)$, the boundary value problem (3.5) is equivalent to

$$v_y \frac{\partial R}{\partial y} + \varepsilon F \frac{\partial R}{\partial v_x} = \frac{1}{\varepsilon} \mathscr{L} R + \mathscr{N} R + \varepsilon^2 \tilde{Q}(R, R) + \varepsilon^3 A$$

$$R(-1, v) = \zeta^-, \quad v_y > 0; \qquad R(1, v) = \beta_R \bar{M}_+(v) + \zeta^+, \quad v_y < 0 \qquad (3.9)$$

$$\int_{\mathbb{R}^3} dv\, v_y R = 0, \qquad y \in [-1, 1]$$

with

$$\mathscr{N} R = \mathscr{L}^1 R + I(R) \left[\sum_{n=2}^{6} \varepsilon^{n-1} \mathscr{L} f_n + \mathscr{L} b_1 - \varepsilon F \frac{\partial M}{\partial v_x} \right]$$

$$\tilde{Q}(R, R) = Q(R, R) + 2I(R)\, \mathscr{L} R$$

$\zeta^\pm = -\sum_{n=1}^{6} \varepsilon^{n-3} \gamma_{n,\varepsilon}^\pm$, and β_R depending on α_R^+ and α_R^-.

Once α_R^- has been chosen as before to satisfy the normalization condition, the parameter β_R is free and can be chosen to satisfy the vanishing net flow condition $(3.9)_3$. Hence we put

$$\beta_R = \int_{v_y > 0} v_y R(1, v) + \int_{v_y < 0} v_y \zeta^+ \qquad (3.10)$$

The solution to the nonlinear boundary value problem (3.9)–(3.10) is constructed using the following iterative procedure. We define R_n, for $n \geqslant 1$, as the solution of the problem (3.9)–(3.10) with $\tilde{Q}(R, R)$ replaced by $\tilde{Q}(R_{n-1}, R_{n-1})$ and $R_0 = 0$. The convergence of the procedure is consequence of good estimates for the linear problem obtained from (3.9)–(3.10) replacing $\varepsilon A + \tilde{Q}(R_{n-1}, R_{n-1})$ with a given source D:

$$v_y \frac{\partial R}{\partial y} + \varepsilon F \frac{\partial R}{\partial v_x} = \frac{1}{\varepsilon} \mathscr{L} R + \mathscr{N} R + \varepsilon^2 D$$

$$R(-1, v) = \zeta^-, \quad v_y > 0; \qquad R(1, v) = \beta_R \bar{M}_+(v) + \zeta^+, \quad v_y < 0 \qquad (3.11)$$

$$\int_{\mathbb{R}^3} dv\, v_y R = 0, \qquad y \in [-1, 1]$$

The solution of the problem (3.11)–(3.10) requires the decomposition into low- and high-velocity parts. Here is the basic modification of the method of ref. 1. For any integer $s \geqslant 1$ we put $P_s(v) = (1+v^2)^{-s/2}$ and

$$R = \sqrt{M}\, g + P_s h \tag{3.12}$$

In ref. 1, instead of P_s there is a Maxwellian $M_*^{1/2}$ with sufficiently high temperature. Here g and h are the *low-velocity* and the *high-velocity* parts of the solution, respectively. They are defined as the solutions of the following linear boundary value problems:

$$\begin{gathered} v_y \frac{\partial g}{\partial y} + \varepsilon F \frac{\partial g}{\partial v_x} + (\mu + \varepsilon F \mu')\,\hat{g} = \varepsilon^{-1} L g + \varepsilon^{-1} \chi_y \sigma^{-1} K_s h + N^1 \hat{g} + \Lambda \hat{g} \\ g(1, v) = \beta_g \bar{M}_+(v)\, M^{-1/2}(1, v), \quad v_y < 0; \qquad g(-1, v) = 0, \quad v_y > 0 \end{gathered} \tag{3.13}$$

$$\begin{aligned} v_y \frac{\partial h}{\partial y} + \varepsilon F \frac{\partial h}{\partial v_x} &+ \varepsilon F \mu_s' h + (\mu + \varepsilon F \mu')\, \sigma(\bar{g} + g_2) \\ &= \varepsilon^{-1}(-v + \bar{\chi}_y K_s)\, h + N_s[\sigma(\bar{g} + g_2)] + h \\ &+ \varepsilon[N_s^{(2)} \hat{g} + \Delta \Lambda \hat{g} + \varepsilon^2 d] \\ h(1, v) &= P_s^{-1}[\zeta^+(v) + \beta_h \bar{M}_+(v)], \qquad v_y < 0 \\ h(-1, v) &= P_s^{-1} \zeta^-(v), \qquad v_y > 0 \end{aligned} \tag{3.14}$$

The notation used in (3.13)–(3.14) is the following. For $\alpha = 0, \ldots, 4$, $\psi_\alpha = \tilde{\psi}_\alpha M^{1/2}$ with $\tilde{\psi}_\alpha$ the collision invariants 1, v_x, v_y, v_z, $v^2/2$, suitably orthonormalized in $L_2(dv)$. The function g_2 is the component of g along ψ_2, $\hat{g} = \sum_{\alpha \neq 2} p_\alpha(y)\, \psi_\alpha$ is the part of g along the other collision invariants, and $\bar{g}$ is the part of g orthogonal to the collision invariants, so that $g = \bar{g} + \hat{g} + g_2$. Here $\chi_y(v)$ is the characteristic function of the set $\{v \in \mathbb{R}^3 \mid v^2 \leqslant y^2\}$ and $\bar{\chi}_y = 1 - \chi_y$. Moreover,

$$\begin{aligned} Lf &= M^{-1/2} 2Q(M, M^{1/2} f) = (-v + K)\, f \\ L_s f &= P_s^{-1} 2Q(M, P_s f) = (-v + K_s)\, f \end{aligned} \tag{3.15}$$

$$\begin{aligned} \beta_g &= \int_{v_y > 0} dv\, v_y M^{1/2} g(1, v) \\ \beta_h &= \int_{v_y > 0} dv\, v_y P_s h(1, v) + \int_{v_y < 0} dv\, v_y \zeta^+ \end{aligned} \tag{3.16}$$

$$\mu = v_y \frac{1}{2} \partial_y \log M, \qquad \mu' = \frac{1}{2} \partial_{v_x} \log M$$
$$\mu'_s = \partial_{v_x} \log P_s, \qquad \sigma_s = \left(\frac{M}{P_s}\right)^{1/2} \tag{3.17}$$

$$d = P_s^{-1} D, \qquad \tilde{b}_n^{\pm} = b_n^{\pm} M_{\pm}^{-1/2}, \qquad N_s f = P_s^{-1} \mathcal{N}(P_s f) \tag{3.18}$$

and

$$N_s^{(2)} \hat{g} = 2P_s^{-1} \left\{ Q\left[\sum_{n=2}^{6} \varepsilon^{n-2} f_n, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})\, M)\right] - 2F\mu' I(M^{1/2}\hat{g})\, M \right\} \tag{3.19}$$

$$N^1 \hat{g} = 2M^{-1/2}\{ Q[B_1, M^{1/2}\hat{g}] + Q[b_1^-, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})\, M)]\}$$
$$A\hat{g} = M^{-1/2} 2Q[\tilde{b}_1^+ M^{1/2}, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})\, M)]$$
$$\Delta A\hat{g} = -P_s^{-1} 2Q[\tilde{b}_1^+ \Delta' M_+, (M^{1/2}\hat{g} + I(M^{1/2}\hat{g})\, M)] \tag{3.20}$$
$$\Delta' M_+ = \varepsilon^{-1}(M^{1/2} - M_+^{1/2})$$

We notice that μ'_s is a bounded function of v for any positive s. In ref. 1 the corresponding function $\mu'_* = \frac{1}{2}\partial_{v_x} \log M_*$ grows linearly in v. This is the main advantage of the decomposition (3.12) compared to the one used in ref. 1. In several situations we will use the following properties of the linear Boltzmann operator L. Here $(\cdot, \cdot)$ is the scalar product in $L_2(\mathbb{R}^2, dv)$. We have

$$(f, [-L]\, f) \geqslant c(\bar{f}, \nu\bar{f}), \qquad \nu_0(1 + |v|)^{\beta} \leqslant \nu(y, v) \leqslant \nu_1(1 + |v|)^{\beta} \tag{3.21}$$

for any $y \in (-1, 1)$ and some suitable positive constant c. This suggests that we consider the weighted L_2-norm

$$\|f\|^2 = \int_{[-1, 1] \times \mathbb{R}^3} dy\, dv\, (1 + |v|)^{\beta} f^2(y, v) \tag{3.22}$$

In particular, the scalar product $(h, \mu'_* h)$ is bounded in this norm when $\beta = 1$ (hard spheres), while it is *not* for $\beta < 1$. On the other hand, the scalar product $(h, \mu'_s h)$ is bounded for any positive β and for any $s > 0$. The estimate of the low-velocity part g is not seriously affected by the replacement of $M_*^{1/2}$ by P_s. In fact next proposition is basically taken from ref. 1.

Proposition 3.2. There exist positive constants $\varepsilon_0 > 0$, $q_0 > 0$, and $C_\gamma > 0$ such that for $\varepsilon < \varepsilon_0$ and $q < q_0$ the solution to Eqs. (3.13) satisfies the bound

$$\|g_2\| + \varepsilon \|\bar{g}\| + \varepsilon^2 \|\hat{g}\| \leqslant C_\gamma \|h\| \tag{3.23}$$

The proof of the proposition is exactly as in ref. 1. The only change is in the estimate of the term $\chi_\gamma \sigma^{-1} K_s h$, which requires the estimates for K_s presented below and used in much more substantial way in the estimate of h. We omit the details. In next section we will prove the following result.

Proposition 3.3. There are $\varepsilon_0 > 0$, $q_0 > 0$, γ_0, and $s_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $q < q_0$, $\gamma > \gamma_0$, and $s > s_0$, the solution h of the problem (3.14) satisfies the bound

$$\|h\| \leqslant c\varepsilon^3 \|d(1 + |v|)^{-\beta}\| + c\varepsilon^{1/2}\{|h_-| + |h_+|\} \tag{3.24}$$

Propositions 3.2 and 3.3 provide L_2 bounds for g and h and hence weighted L_2 bounds for R. Using the regularizing properties of the inverse of the transport operator and of the operator K, it is possible to enhance such bounds to weighted L_∞ bounds for g, h, and R. This follows exactly as in ref. 2; we omit the details. In conclusion we have proved the following result.

Theorem 3.4. There are $\varepsilon_0 > 0$, $q_0 > 0$, γ_0, and $s_0 > 0$ such that for $\varepsilon < \varepsilon_0$, $q < q_0$, $\gamma > \gamma_0$, and $s > s_0$, the solution of the boundary value problem (3.11)–(3.10) satisfies the bound

$$|R|_s \leqslant c\varepsilon^{1/2} |D|_{s-\beta} + c\varepsilon^{-2}(|\zeta^-|_s + |\zeta^+|_s) \tag{3.25}$$

The above theorem is all we need to prove the convergence of the sequence R_n and hence Theorem 2.1. We note that the above theorem also implies uniqueness of the solution f^ε in the class of the functions f such that $\varepsilon^{-3+\zeta} |f - M - \varepsilon f_1 - \varepsilon^2 f_2|_p$ is bounded uniformly in ε for $\zeta < 1/2$ and $p > p_0$. We refer to ref. 1 for details.

4. PROOF OF PROPOSITION 3.3

For the proof of Proposition 3.3 we will use the following estimates for the collision operator. We decompose the collision operator into gain loss parts as $Q(f, g) = J(f, g) + J(g, f) - gS(f)$. In particular, we will be interested in bounds for

$$K_s f = P_s^{-1}[J(M, P_s f) + J(P_s f, M) - MS(P_s f)]$$

Let L^q_s be the space of the measurable real functions $f(v)$ on $\mathbb{R}^3$ such that $(1+v^2)^{s/2} f(v)$ is in $L_q(\mathbb{R}^3, dv)$ with norm

$$\|f\|^q_{q,s} = \int_{\mathbb{R}^3} dv\, |f(v)|^q (1+v^2)^{qs/2}$$

Proposition 4.1. Given g, let $Hf = |J(f,g)| + |J(g,f)| + |g| \cdot |S(f)|$. Then there are constants C and C_s and a function $\delta(s) \to 0$ as $s \to \infty$ such that

$$\|(1+|v|)^{-\beta} \bar{\chi}_y H(f)\|_{2,s} \leqslant \delta(s) \|g\|_1 \|f\|_{2,s} + C(1+y)^{-\beta/2} [\|g\|_1 \|f\|_{2,s+\beta/2} + (\|g\|_{2,s+4} + \|g\|_{2,s+\beta})] \|f\|_{2,s} \tag{4.1}$$

In particular, for $g = M$ and f replaced by $P_s f$, recalling (3.22), it follows that

$$\|(1+|v|)^{-\beta} \bar{\chi}_y K_s f\|_{2,0} \leqslant C\delta(s) \|f\| + C_s(1+y)^{-\beta/2} \|f\| \tag{4.2}$$

Before giving the proof of Proposition 4.1, we conclude the proof of Proposition 3.3. The argument is close to the one of ref. 2. Below we use the notation $\langle f \rangle = \int_{\mathbb{R}^3} dv\, f(v)$. Multiply Eq. $(3.14)_1$ by h and integrate on $[-1, 1] \times \mathbb{R}^3$. After a few integrations by parts one is left with

$$\mathscr{I} + \int dy \langle \varepsilon F \mu'_s h^2 \rangle + \int dy \langle (\mu + \varepsilon F \mu')\, h \sigma(\bar{g} + g_2) \rangle$$

$$= \int dy\, dv\, h\{\varepsilon^{-1}(-v + \bar{\chi}_y K_s)\, h + N_s[\sigma(\bar{g} + g_2) + h] + \varepsilon(N_s^{(2)} \hat{g} + \Delta\Lambda \hat{g}) + \varepsilon^2 d\} \tag{4.3}$$

with $\mathscr{I} = \langle v_y h^2(1, v) \rangle - \langle v_y h^2(-1, v) \rangle$. Using the boundary conditions for h, we have

$$\int_{v_y<0} |v_y|\, h^2(-1, v) + \int_{v_y>0} |v_y|\, h^2(1, v) + \leqslant \mathscr{I} + c(\beta_h^2 + |h_+|^2 + |h_-|^2) \tag{4.4}$$

It can be checked as in ref. 1 that

$$\beta_h \leqslant c(\varepsilon^{1/2} \|v^{-1}[Z + \varepsilon^{-1} \bar{\chi}_y K_s h]\| + |h_-| + |h_+|) \tag{4.5}$$

with

$$
\begin{aligned}
Z &= -\varepsilon F\mu_s' h + (\mu + \varepsilon F\mu')\,\sigma(\bar{g} + g_2) \\
&\quad - N_s[\sigma(\bar{g} + g_2) + h] + \varepsilon[N_s^{(2)}\hat{g} + \Delta\Lambda\hat{g}] + \varepsilon^2 d \\
h_{\pm} &= \zeta^{\pm} P_s^{-1} \\
|h_{\pm}| &= \sup_{v_y \lessgtr 0} |h_{\pm}(v)|
\end{aligned} \tag{4.6}
$$

Therefore

$$\mathscr{I} \geqslant -c(\varepsilon\, \|\nu^{-1}[Z + \varepsilon^{-1}\bar{\chi}_\gamma K_s h]\| + |h_-|^2 + |h_+|^2) \tag{4.7}$$

Using this bound in (4.3) and the estimate $(3.21)_2$, we obtain

$$\nu_0\, \|h\|^2 \leqslant c\varepsilon^2(\|\nu^{-1}[Z + \varepsilon^{-1}\bar{\chi}_\gamma K_s h]\|^2 + \varepsilon(|h_-|^2 + |h_+|^2) \tag{4.8}$$

One can use (4.1) and Proposition 3.2 to get the bound, for ε small enough,

$$\varepsilon^2\, \|\nu^{-1} Z\|^2 \leqslant c_\gamma(q\, \|h\|^2 + \varepsilon^6\, \|\nu^{-1} d\|^2)$$

where the constant c_γ depends on γ. Using (4.2), we get

$$\|h\|^2\, \{\nu_0 - \delta(s)\, c_1 - C(1+\gamma)^{-\beta} - c_\gamma q\} \leqslant \varepsilon^6\, \|\nu^{-1} d\|^2 + \varepsilon(|h_-|^2 + |h_+|^2) \tag{4.9}$$

We choose s such that $c_1 \delta(s) \leqslant \nu_0/4$, then γ such that $C(1+\gamma)^{-\beta} \leqslant \nu_0/4$, and finally q such that $c_\gamma q \leqslant \nu_0/4$. With these choices we conclude that

$$\|h\| \leqslant c\varepsilon^3\, \|\nu^{-1} d\| + \varepsilon^{1/2}(|h_-| + |h_+|) \tag{4.10}$$

and hence Proposition 3.3 is proved.

Proof of Proposition 4.1. In ref. 3 an L_∞ estimate similar to the one of Proposition 4.1 is proved. Therefore, by interpolation theorems, it is enough to get an L_1 bound. To this end, we introduce

$$\bar{J}(f, g)(v) = \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} f(v')\, g(v_1')\, \sin\theta \cos\theta\, |v - v_1|^\beta\, d\theta\, d\varphi\, dv_1 \tag{4.11}$$

The quantities we have to estimate are in fact bounded by $\bar{J}(f, g)$. We only discuss the estimate of the term

$$\left\| \frac{1}{(1 + |v|)^\beta} \bar{\chi}_\gamma \bar{J}(f, g) \right\|_{1, s}$$

while the others are obtained with simpler arguments. We divide the integration domain for the velocity into three parts:

$$A = \{ |v'| < \delta\, |v| \text{ and } |v_1| < \eta\, |v| \}$$

$$B = \{ |v'| < \delta\, |v| \text{ and } |v_1| \geqslant \eta\, |v| \}$$

$$C = \{ |v'| \geqslant \delta\, |v| \text{ with } \delta \leqslant \eta \}$$

Moreover, we denote by $\bar{J}_A$ the part of $\bar{J}$ corresponding to the integration domain A and so on. In A the following estimates hold:

$$\begin{aligned} |v - v_1'| &< (\eta + \delta)\, |v| \\ |v_1'| &> (1 - \delta)\, |v| \\ |v - v'| &\geqslant (1 - \delta)\, |v_1| \\ |v - v_1|^\beta &< (1 + \eta)^\beta\, |v|^\beta \end{aligned} \tag{4.12}$$

From $(4.12)_{1,2}$ it follows that the range of θ is the interval $[0, \theta_{\max}]$ with $\theta_{\max} \leqslant (\eta + \delta)/(1 - \delta) < 3\eta$, after choosing δ small enough. We use the notation

$$h(\theta) = |\sin(\theta) \cos(\theta)|, \qquad \rho_h(\eta) = \int_0^\eta h(\theta)\, d\theta$$

Moreover, from $(4.12)_{1,2,4}$, we obtain

$$\frac{1}{(1 + |v|)^\beta} \frac{\bar{\chi}_\gamma\, |v - v_1|^\beta\, P_s^{-1}(v)}{P_s^{-1}(v_1')} \leqslant \frac{(1 + \eta)^\beta}{(1 - \delta)^s}$$

The change of variables $dv\, dv_1 \to dv'\, dv_1'$ then provides the bound

$$\begin{aligned} &\left\| \frac{1}{(1 + |v|)^\beta} \bar{\chi}_\gamma \bar{J}_A(g, f) \right\|_{1,s} \\ &\quad \leqslant \frac{C}{(1 - \delta)^s} \rho_h(3\eta) \cdot \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} g(v')\, f(v_1')\, P_s^{-1}(v_1')\, dv\, dv_1 \\ &\quad = \frac{C}{(1 - \delta)^s} \rho_h(3\eta)\, \|g\|_1 \cdot \|f\|_1 \end{aligned} \tag{4.13}$$

To estimate $\bar{J}_B$, we note that $|v-v_1|^\beta = |v'-v|^\beta (\cos\theta)^{-\beta}$. The energy conservation and the definition of the set B imply $v_1'^2 = v_1^2 + v^2 - v'^2 > (1+\eta^2-\delta^2)\, v^2$. Hence

$$\frac{\bar{\chi}_\gamma P^{-1}_{s+\beta/2}(v)}{P^{-1}_{s+\beta/2}(v_1')} \leqslant \frac{1}{(1+\eta^2-\delta^2)^{(s+\beta/2)/2}}$$

Again the change of variables $dv\, dv_1 \to dv'\, dv_1'$ then provides the bound

$$\left\| \frac{1}{(1+|v|)^\beta} \bar{\chi}_\gamma \bar{J}_B(g, f) \right\|_{1,s} \leqslant \frac{1}{(1+\gamma)^{\beta/2}} \frac{C}{(1+\eta^2-\delta^2)^{s+\beta/2}} \|g\|_1 \|f\|_{1,(s+\beta)/2} \tag{4.14}$$

In the set C we have $P_s(v')\, P_s(v)^{-1} \leqslant C\delta^{-s}$. Multiplying and dividing by $P_r(v')$ for some $r>0$ and using the Holder inequality, we have

$$\begin{aligned} &\left\| \frac{1}{(1+|v|)^\beta} \bar{\chi}_\gamma (1+v^2)^{s/2} \bar{J}_C(g, f) \right\|_q^q \\ &\leqslant \frac{C}{\delta^{sq/2}} \frac{1}{(1+\gamma)^{\beta q/2}} \int_{\mathbb{R}^3} |v'|^{qs} \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} g(v')\, P^{-1}_{rq}(v') \\ &\quad \times f(v_1')\, P^{-1}_{sq}(v_1')\, h(\theta)\, d\theta\, d\varphi\, dv_1 \\ &\quad \times \left(\int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi/2} \frac{h(\theta)}{P^{-1}_{(r-\beta/2)q'}(v')\, P^{-1}_{(s-\beta)q'}(v-v_1')}\, d\theta\, d\varphi\, dv_1 \right)^{q/q'} dv \end{aligned} \tag{4.15}$$

The last integral converges for $\min(r, s) \geqslant \beta + 3(1-1/q)$. Hence we choose $r=4$ and below s is assumed to satisfy the bound.

Combining estimates (4.13)–(4.15), we get

$$\begin{aligned} &\left\| \frac{1}{(1+|v|)^\beta} \bar{\chi}_\gamma \bar{J}(f, g) \right\|_{1,s} \\ &\leqslant \rho_h(3\eta) \frac{C}{(1-\delta)^s} \|g\|_1 \|f\|_{1,s} \\ &\quad + C \frac{1}{(1+\gamma)^{\beta/2}} \left(\frac{1}{(1+\eta^2-\delta^2)^{(1/2)(s+\beta/2)-1}} \|g\|_1 \|f\|_{1,s+\beta/2} \right. \\ &\quad \left. + [\|g\|_{1,4+s} + \|g\|_{1,s+\beta}]\, \|f\|_{1,s} \right) \end{aligned}$$

We choose $\eta = (\sqrt{s})^{-1}$ and $\delta = s^{-1}$. Therefore $\delta^2 \leqslant \eta^2/2 = 1/(2s)$. Hence

$$(1 + \eta^2 - \delta^2)^{+1-(1/2)(s+\beta/2)} \left(1 + \frac{\eta^2}{2}\right)^{+1-(1/2)(s+\beta/2)} \leqslant C$$

Moreover, $\rho_h(3\eta) \leqslant c\eta^2$. Then

$$\frac{\rho_h(3\eta)}{(1-\delta)^s} \leqslant c \frac{1}{s} \frac{1}{(1-1/s)^s} < cs^{-1}$$

This proves Proposition 4.1 with $\delta(s) = s^{-1}$.

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